Section 5.2 (16, 36, 76, 80, 82)

16. (a) $\langle f, g\rangle = \int_{-1}^{1} (-1)(1 - 2x^2)dx = \left[\frac{2x^3}{3} - x\right]_{-1}^{1} = \left(\frac{2}{3} - 1\right) - \left(-\frac{2}{3} + 1\right) = -\frac{2}{3}$

(b) $\|f\|^2 = \langle f, f\rangle = \int_{-1}^{1} (-1)^2 dx = x\Big]_{-1}^{1} = 2$

$\|f\| = \sqrt{2}$

(c) $\|g\|^2 = \langle g, g\rangle = \int_{-1}^{1} (1 - 2x^2)^2 dx = \int_{-1}^{1} (1 - 4x^2 + 4x^4)dx$

$= \left[x - \frac{4x^3}{3} + \frac{4x^5}{5}\right]_{-1}^{1} = \left(1 - \frac{4}{3} + \frac{4}{5}\right) - \left(-1 + \frac{4}{3} - \frac{4}{5}\right) = \frac{14}{15}$

$\|g\| = \sqrt{\frac{14}{15}} = \frac{\sqrt{210}}{15}$

(d) Use the fact that $d(f, g) = \|f - g\|$. Because $f - g = -1 - (1 - 2x^2) = 2x^2 - 2$, you have

$\langle f - g, f - g\rangle = \int_{-1}^{1} (2x^2 - 2)^2 dx = \int_{-1}^{1} (4x^4 - 8x^2 + 4)dx = \left[\frac{4x^5}{5} - \frac{8x^3}{3} + 4x\right]_{-1}^{1} = \frac{64}{15}.$

$d(f, g) = \sqrt{\frac{64}{15}} = \frac{8\sqrt{15}}{15}$

36. The product $\langle \mathbf{u}, \mathbf{v}\rangle$ is not an inner product because Axiom 2 is not satisfied. For example, let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (1, 2)$, $\mathbf{w} = (2, 0)$ (Thus $\mathrm{v} + \mathrm{w} = (3, 2)$).

Axiom 2: Then $\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = (1)(1) + (3)(2) = 7$. Which does not equal

$\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle = [(1)(1) + (1)(2)] + [(1)(1) + (2)(0)] = 4.$

76. $\langle f, g\rangle = \int_{-\pi}^{\pi} sin2x\ cos2x\ dx = \int_{0}^{0} \frac{1}{2} u du = 0$ (where u = sin2x). Hence $\mathrm{proj}_g f = 0$.

80. (a) False. The norm of a vector **u** is defined as a square root of $\langle \mathbf{u}, \mathbf{u}\rangle$.

(b) False. The angle between $a\mathbf{v}$ and $\mathbf{v}$ is zero if $a > 0$ and it is π if $a < 0$.

82. $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle$

$= (\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle) + (\langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle)$

$= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$

Section 5.3 (10, 31, 48, 63)

10. The set is *not* orthogonal because

$$\left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right) \cdot \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) = \frac{\sqrt{10}}{30} \neq 0.$$

31. Because $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, the given vectors are orthogonal. Normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{3}(1, -2, 2) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{3}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{3}(2, -1, -2) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

So, the orthonormal basis is $\left\{\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\right\}$.

48. (a) True. See definition on page 306.

(b) True. See Theorem 5.10 on page 309.

(c) True. See page 312.

63. For $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ an orthonormal basis for R^n and $\mathbf{v}$ any vector in R^n,

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2\rangle \mathbf{u}_2 + \ldots + \langle \mathbf{v}, \mathbf{u}_n\rangle \mathbf{u}_n$$

$$\|\mathbf{v}\|^2 = \left\|\langle \mathbf{v}, \mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2\rangle \mathbf{u}_2 + \ldots + \langle \mathbf{v}, \mathbf{u}_n\rangle \mathbf{u}_n\right\|^2.$$

To simplify notation let $w_i = \langle \mathbf{v}, \mathbf{u_i}\rangle = \mathbf{v} \cdot \mathbf{u_i}$ (a scalar).
Thus $\|\mathbf{v}\|^2 = \|w_1\mathbf{u_1} + w_2\mathbf{u_2} + \cdots + w_n\mathbf{u_n}\|$
$= \langle w_1\mathbf{u_1} + w_2\mathbf{u_2} + \cdots + w_n\mathbf{u_n},\ w_1\mathbf{u_1} + w_2\mathbf{u_2} + \cdots + w_n\mathbf{u_n}\rangle$
Now since $\langle \mathbf{u_i}, \mathbf{u_j}\rangle = 0$ whenever $i \neq j$ all the cross terms go to zero and we get
$\|\mathbf{v}\|^2 = w_1^2\langle \mathbf{u_1}, \mathbf{u_1}\rangle + w_2^2\langle \mathbf{u_2}, \mathbf{u_2}\rangle + \cdots + w_n^2\langle \mathbf{u_n}, \mathbf{u_n}\rangle$. But $\langle \mathbf{u_i}, \mathbf{u_i}\rangle = 1$ so
$\|\mathbf{v}\|^2 = w_1^2 + w_2^2 + \cdots + w_n^2$. But $w_i = \mathbf{v} \cdot \mathbf{u_i}$, so adding an unnecessary absolute value (it's a square), we get $\|\mathbf{v}\|^2 = |\mathbf{v} \cdot \mathbf{u_1}|^2 + |\mathbf{v} \cdot \mathbf{u_2}|^2 + \cdots + |\mathbf{v} \cdot \mathbf{u_n}|^2$

END HW 4 Begin HW 5

Section 6.1 (12, 22, 26, 32, 37, 50)

12. T is *not* a linear transformation because it does not preserve addition. For example,

$$T(1, 1, 1) + T(1, 1, 1) = (2, 2, 2) + (2, 2, 2) = (4, 4, 4),$$

but $T(2, 2, 2) = (3, 3, 3)$.

22. T preserves addition.

$$T(a_0 + a_1x + a_2x^2) + T(b_0 + b_1x + b_2x^2) = (a_1 + 2a_2x) + (b_1 + 2b_2x)$$
$$= (a_1 + b_1) + 2(a_2 + b_2)x$$
$$= T((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2)$$

T preserves scalar multiplication.

$$T(c(a_0 + a_1x + a_2x^2)) = T(ca_0 + ca_1x + ca_2x^2)$$
$$= ca_1 + 2ca_2x = c(a_1 + 2a_2x) = cT(a_0 + a_1x + a_2x^2)$$

Therefore, T *is* a linear transformation.

26. Because $(-2, 4, -1)$ can be written as

$$(-2, 4, -1) = -2(1, 0, 0) + 4(0, 1, 0) - 1(0, 0, 1),$$

you can use Property 4 of Theorem 6.1 to write

$$T(-2, 4, -1) = -2T(1, 0, 0) + 4T(0, 1, 0) - T(0, 0, 1)$$
$$= -2(2, 4, -1) + 4(1, 3, -2) - (0, -2, 2)$$
$$= (0, 6, -8).$$

32. Because the matrix has 2 columns, the dimension of R^n is 2.
Because the matrix has 3 rows, the dimension of R^m is 3. So, $T: R^2 \rightarrow R^3$.

37. (a) $T(2, 4) = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 4 \end{bmatrix} = (10, 12, 4)$

(b) The preimage of $(-1, 2, 2)$ is given by solving the equation

$$T(v_1, v_2) = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

for $\mathbf{v} = (v_1, v_2)$. Since (-1, 0) is the solution of

$$\begin{aligned} v_1 + 2v_2 &= -1 \\ -2v_1 + 4v_2 &= 2 \\ -2v_1 + 2v_2 &= 2 \end{aligned}$$

we know $(-1, 0)$ is the preimage of $(-1, 2, 2)$ under T.

(c) Because the system of linear equations represented by the equation

$$\begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution, $(1, 1, 1)$ has no preimage under T.

50. If $D_x(g(x)) = \frac{1}{x}$, then $g(x) = \ln x + C$.

Section 6.2 (24, 34, 36, 46)

24. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So, $\ker(T) = \{(t, -t, s, -s) : s, t \in R\}$.

(b) $\text{nullity}(T) = \dim(\ker(T)) = 2$

(c) Transpose A and find its equivalent row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ So, $\text{range}(T) = R^2$.

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

34. Because $\text{rank}(T) + \text{nullity}(T) = 3$, and you are given $\text{rank}(T) = 3$, then $\text{nullity}(T) = 0$. So, the kernel of T is the single point $\{(0, 0, 0)\}$, and the range is all of R^3.

36. The kernel of T is determined by solving $T(x, y, z) = (-x, y, z) = (0, 0, 0)$, which implies that the kernel is the single point $\{(0, 0, 0)\}$. Since $\text{rank}(T) + \text{nullity}(T) = 3$, the rank of T is 3. So, the range of T is all of R^3.

46. The equation $T(p) = \int_0^1 p(x)dx = \int_0^1 (a_0 + a_1x + a_2x^2)dx = 0$ yields $a_0 + a_1/2 + a_2/3 = 0$.

We can eliminate one of these three constants. For example:

Letting $a_2 = -3b$, $a_1 = -2a$, you have $a_0 = -a_1/2 - a_2/3 = a + b$,

and $\ker(T) = \{(a + b) - 2ax - 3bx^2 : a, b \in R\}$.